Lecture 7: Damped and Driven Oscillations
• Last time, we found for underdamped oscillators:
  \[ x(t) = e^{-\beta t} \left[ (A_1 + A_2) \cos \omega_1 t + i (A_1 - A_2) \sin \omega_1 t \right] \]
  
  • A\textsubscript{1} and A\textsubscript{2} are complex numbers, but our answer must be real
    – Implies that A\textsubscript{1} and A\textsubscript{2} are complex conjugates
    – Can write them as: \[ A_1 = Ae^{i\delta} \quad A_2 = Ae^{-i\delta} \]
  
  • We now have:
    \[ x(t) = e^{-\beta t} \left[ A(\cos \delta + i \sin \delta + \cos \delta - i \sin \delta) \cos \omega_1 t \right. \]
    \[ \quad + iA(\cos \delta + i \sin \delta - \cos \delta + i \sin \delta) \sin \omega_1 t \left. \right] \]
    \[ = Ae^{-\beta t} \left[ 2 \cos \delta \cos \omega_1 t - 2 \sin \delta \sin \omega_1 t \right] \]
    \[ = 2Ae^{-\beta t} \cos(\omega_1 t + \delta) \]

  • Since we can always redefine the constant A to get rid of the 2 in front of the equation, the general solution is:
    \[ x(t) = Ae^{-\beta t} \cos(\omega_1 t + \delta) \]
Properties of underdamped motion

• An underdamped system still oscillates:

Note, though, that the motion is not periodic – it never returns to the same point with the same velocity as before

• The quantity $\omega_1$ can still be related to the time interval between crossings of the $x$ axis

• For light damping, $\omega_1$ is very close to $\omega_0$
Underdamped Motion in Phase Space

- Since the motion is not periodic, we no longer get closed loops. In addition to amplitude, path depends on $\beta$:

$$\omega_0 = 0.5, \ A = 100$$

$\beta = 0.05$

$\beta = 0.1$

$\beta = 0.25$
More Damping

• If the damping parameter is large enough that $\sqrt{\beta^2 - \omega_0^2} = 0$ the system is called “critically damped”

• In this case expressions of the form $te^{-\beta t}$ also satisfy the equation of motion, so the general solution is:

$$x(t) = (A + Bt)e^{-\beta t}$$

• From this we see that $A$ is the initial position and $B-\beta A$ is the initial velocity

• In this case the solutions do not oscillate, but can cross the $x$-axis once if there is a large initial velocity toward equilibrium
• The motion may look like any of the following:

  ![Initial velocity negative](image)
  ![Initial velocity zero](image)
  ![Initial velocity positive](image)

• No matter what the initial conditions are, the system settles to within a given distance of equilibrium faster with critical damping than with any other choice of damping parameter
  – Automobile shock absorbers, for example, should be critically damped
Overdamped Motion

• If $\beta$ is even larger the system is *overdamped*
• The quantity $\omega_2 \equiv \sqrt{\beta^2 - \omega_o^2}$ is real, so the position as a function of time is given by:

$$x(t) = e^{-\beta t} \left[ A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t} \right]$$

• Features:
  – No hint of oscillatory motion here ($\omega_2$ can’t be interpreted as an angular frequency)
  – Position always approaches equilibrium for large $t$
    • But not as quickly as a critically-damped system would
  – System can cross $x = 0$ once (as in critically damped case)
Driven Oscillations

- There are many examples in which an external agent applies a force to an oscillator
  - Sometimes essential to intended function (e.g., a radio), sometimes an annoyance (e.g., wind gusts hitting a skyscraper)
- This external force can have any form, but we’ll consider the particular case of a sinusoidal force:

  \[ F = F_0 \sin \omega t \]

  This \( \omega \) can be anything we choose – it’s not related to the natural oscillation frequency \( \omega_0 \)

- This means the equation of motion is:

  \[ m\ddot{x} + b\dot{x} + kx = F_0 \sin \omega t \]
• After dividing through by \( m \) and redefining the constants, this becomes:

\[
\ddot{x} + 2\beta \dot{x} + \omega^2 x = A \sin \omega t
\]

• This is known as a “linear inhomogeneous equation”

• To solve it, let’s assume that the solution has a form similar to what appears on the right-hand side:

\[
x(t) = C \sin(\omega t + \delta)
\]

• Substituting this into the equation of motion gives:

\[
-C\omega^2 \sin(\omega t + \delta) + 2C \beta \omega \cos(\omega t + \delta) + \omega^2 C \sin(\omega t + \delta) = A \sin \omega t
\]
• Expanding gives:

\[-C \omega^2 [\cos \omega t \sin \delta + \cos \delta \sin \omega t] + 2C \beta \omega [\cos \omega t \cos \delta - \sin \omega t \sin \delta] + \omega^2 C [\cos \omega t \sin \delta + \cos \delta \sin \omega t]\]

\[= A \sin \omega t\]

\[
\cos \omega t \left[-C \omega^2 \sin \delta + 2C \beta \omega \cos \delta + \omega^2 C \sin \delta \right] + \sin \omega t \left[-A - C \omega^2 \cos \delta - 2C \beta \omega \sin \delta + \omega^2 C \cos \delta \right] = 0
\]

• The only way this can be true for all \(t\) is if \(C\) and \(\delta\) are chosen such that both terms in [] are zero
• Starting with the cosine term, we need:

\[ 2\beta\omega \cos \delta + (\omega_o^2 - \omega^2) \sin \delta = 0 \]

\[ \tan \delta = \frac{-2\beta\omega}{(\omega_o^2 - \omega^2)} \]

• From this, we can determine \( \sin \delta \) and \( \cos \delta \), and then find \( C \):

\[ A - C(-\omega^2 \cos \delta - 2\beta\omega \sin \delta + \omega_o^2 \cos \delta) = 0 \]

\[ C = \frac{A}{(\omega_o^2 - \omega^2) \cos \delta + 2\beta\omega \sin \delta} \]

\[ = \frac{A}{(\omega_o^2 - \omega^2) \sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2} + 2\beta\omega \sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}} \]

\[ = \frac{A}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}} \]
• Putting all of this together, we have the solution:
\[
x(t) = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \sin \left( \omega t - \tan^{-1} \left[ \frac{2\beta\omega}{(\omega_0^2 - \omega^2)} \right] \right)
\]

• Looks great, but you may notice a problem – there’s no “freedom” here (recall that \( A = F_o/m \))
  – and surely the motion depends somehow on initial conditions, doesn’t it?

• To see where these come in, consider what happens if we add a term to our solution:
  \[
x'(t) = x(t) + x_c(t)
\]

  where:
  \[
x_c'' + 2\beta x_c' + \omega_o^2 x_c = 0
\]
• This new function also satisfies the equation of motion:

\[
\ddot{x}' + 2\beta \dot{x}' + \omega_o^2 x' = (\ddot{x} + \ddot{x}_c) + 2\beta (\dot{x} + \dot{x}_c) + \omega_o^2 (x + x_c)
\]

\[
= \ddot{x} + 2\beta \dot{x} + \omega_o^2 x + \ddot{x}_c + 2\beta \dot{x}_c + \omega_o^2 x_c
\]

\[
= A \sin \omega t + 0
\]

• But the equation that \( x_c(t) \) satisfies is just the equation for an undriven oscillator
  – So all the solutions we’ve already explored are part of the solution for driven oscillators as well

• Linear inhomogeneous differential equations in general have this property
  – Solution is the sum of a particular solution that depends on the right-hand side of the equation and a complementary solution that gives zero on the right-hand side
• Some other features of the solution:
  1. The complementary function goes as $e^{-\beta t}$
  2. The initial conditions affect only the complementary solution, not the particular solution
• Both of these facts tell us that the complementary solution gives transient effects

After a long time has passed, the oscillator will move as described by the particular solution, no matter what the initial conditions are
Resonance

- The amplitude attained by a driven oscillator depends strongly on the driving frequency
- The maximum occurs at the “resonance frequency”:

\[
\left. \frac{dA}{d\omega} \right|_{\omega_R} = - \frac{1}{2} \frac{-4\omega_R \left( \omega_o^2 - \omega_R^2 \right) + 8\beta^2 \omega_R}{\left[ \left( \omega_o^2 - \omega_R^2 \right)^2 + 4\beta^2 \omega_R^2 \right]^{3/2}} = 0
\]

\[
\omega_o^2 - \omega_R^2 = 2\beta^2
\]

\[
\omega_R^2 = \omega_o^2 - 2\beta^2
\]

- The sharpness of the resonance depends on the strength of the damping
Q Factor

- Actually, it’s the ratio of the resonance frequency to the damping parameter that determines the sharpness of the resonance. We define:

\[ Q = \frac{\omega_R}{2\beta} = \frac{\sqrt{\omega_o^2 - 2\beta^2}}{2\beta} \]