Lecture 9: More on Calculus of Variations

• Last lecture, we were trying to find the shortest path between two points
• Meant we needed to find a function $y(x)$ that minimizes an integral of the form:

$$J = \int_{x_1}^{x_2} f\{y, y'; x\} dx$$

• We can write any function $y$ as:

$$y(\alpha, x) = y_m(x) + \alpha \eta(x)$$

where $y_m$ is the function that minimizes the integral
• Thus we can solve the problem by setting:

$$\left. \frac{dJ}{d\alpha} \right|_{\alpha=0} = 0$$
• Using the chain rule (and the fact that the limits of integration are constant), we have:

\[
\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' \right) dx
\]

• It would be nice if we could factor out the \( \eta \) dependence

• Try integration by parts on the second term in the integral:

\[
\int u dv = uv - \int v du
\]

\[
u = \eta' dx
\]

\[
\frac{du}{dy'} = \frac{df}{dy'} dx;
\]

\[
t = \eta
\]

\[
\int_{x_1}^{x_2} \frac{df}{dy'} \eta' dx = \left. \frac{df}{dy'} \eta \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \frac{df}{dy'} \eta dx
\]

This term vanishes because \( \eta \) is 0 at both \( x_1 \) and \( x_2 \).
• So we’re left with:

\[
\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \eta - \frac{d}{dx} \frac{\partial f}{\partial y'} \eta \right)dx
\]

\[
= \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta dx
\]

• Now, for the function that minimizes \( J \), we want the above expression to be zero when \( \alpha = 0 \)
  - We get the desired result if:

\[
\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0
\]

This is known as Euler’s equation for determining the function \( y \)
Does it Really Work?

• Let’s return now to the problem we started with – the shortest path between two points in a plane.

• In that case,

\[ f = \sqrt{1 + y'^2} \]

\[ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 - \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) = 0 \]

• This implies that:

\[ \frac{y'}{\sqrt{1 + y'^2}} = \text{const} = A \]

\[ y' = A \sqrt{1 + y'^2} \]

\[ y'^2 = A^2 \left(1 + y'^2\right) \]

\[ y' = \frac{A}{\sqrt{1 - A^2}} = A' \]
• We can now integrate to find:

\[ y(x) = A'x + B \]

Note that \( A' \) and \( B \) are determined by the requirements that \( y(x_1) = y_1 \) and \( y(x_2) = y_2 \)

• It’s a line!
  – …just like we knew all along

• Though the answer in this case was obvious, Euler’s Equation lets us find other minimizing functions that we might not have been able to guess beforehand
The Second Form of Euler’s Equation

• First, a reminder of some multivariate calculus:

\[
\frac{d}{dx} \left( g \{ y_1, y_2, \ldots, y_N : x \} \right) = \sum_{i=1}^{N} \frac{\partial}{\partial y_i} \frac{dy_1}{dx} + \frac{\partial g}{\partial x}
\]

• With this rule, we find:

\[
\frac{d}{dx} \left( f \{ y, y' ; x \} \right) = \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} + \frac{\partial f}{\partial x}
\]

\[
= y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}
\]

\[
y'' \frac{\partial f}{\partial y'} = \frac{df}{dx} - y' \frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}
\]
• We also can use the multiplication rule for derivatives to write:

\[
\frac{d}{dx} \left( y' \frac{\partial f}{\partial y'} \right) = y'' \frac{\partial f}{\partial y'} + y' \frac{d}{dx} \frac{\partial f}{\partial y'}
\]

• Substituting in the expression for \( y'' \frac{\partial f}{\partial y'} \) that was derived on the previous slide gives:

\[
\frac{d}{dx} \left( y' \frac{\partial f}{\partial y'} \right) = \frac{df}{dx} - y' \frac{\partial f}{\partial y} - \frac{\partial f}{\partial x} + y' \frac{d}{dx} \frac{\partial f}{\partial y'}
\]

\[
= \frac{df}{dx} - \frac{\partial f}{\partial x} - y' \left( \frac{\partial f}{\partial y} + \frac{d}{dx} \frac{\partial f}{\partial y'} \right)
\]

This is the left-hand side of Euler’s equation.
• So, if \( y \) satisfies Euler’s equation (which means it’s the function that minimizes the integral we’re considering), the second term must be zero. We’re left with:

\[
\frac{d}{dx} \left( y' \frac{\partial f}{\partial y'} \right) = \frac{df}{dx} - \frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial x} - \frac{d}{dx} \left( f - y' \frac{\partial f}{\partial y'} \right) = 0
\]

• This is the “second form” of Euler’s equation, and at first glance it’s not very useful
  – It’s more complicated than the first form!
• But if \( f \) has no explicit dependence on \( x \) (so \( \frac{\partial f}{\partial x} = 0 \)), we have:

\[
\frac{d}{dx} \left( f - y' \frac{\partial f}{\partial y'} \right) = 0
\]

\[
f - y' \frac{\partial f}{\partial y'} = \text{const.} = C
\]

So we can find \( y \) without taking any full derivatives.

• For example, in our “shortest distance” problem, we had:

\[
f = \sqrt{1 - y'^2}
\]
Since this has no explicit $x$ dependence, we can find the solution via:

$$f - y' \frac{\partial f}{\partial y'} = \sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}} = C$$

$$1 + y''^2 - y'^2 = C \sqrt{1 + y'^2}$$

$$C^2 \left(1 + y'^2\right) = 1$$

$$y'^2 = \frac{1}{C^2} - 1 = C'^2$$

$$y' = C'$$

$$y = C'x + B$$